Topological quantum field theory: Caught with their pants down

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December 28, 2024

Time evolution of quantum states

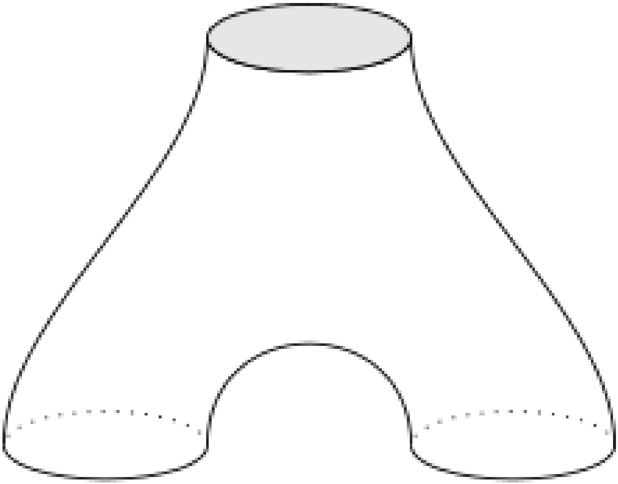
Take a quantum state ψ_{in} at a certain time, a quantum state ψ_{out} at a later time and a unitary operator $\widehat{U}$ describing time evolution.

Chance of ψ_{in} ending in ψ_{out}:

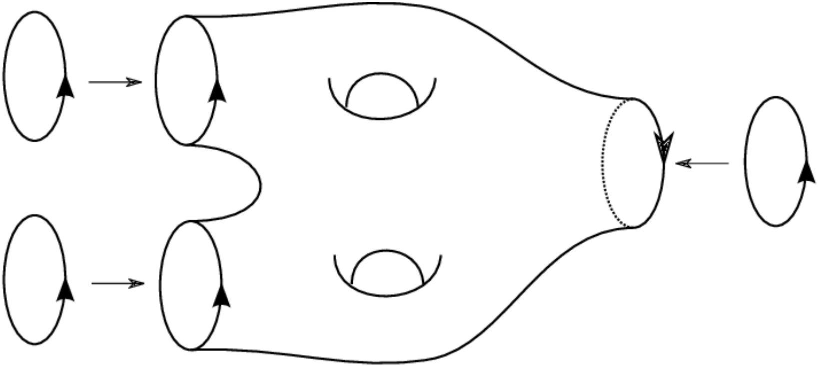
$$|\langle\psi_{\text{in}}|\widehat{U}|\psi_{\text{out}}\rangle|^2 \leq 1$$

Cobordisms

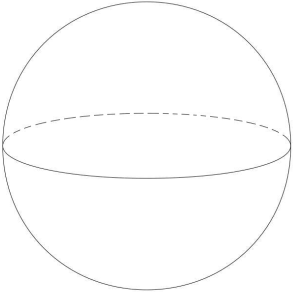
Cobordisms



Cobordisms

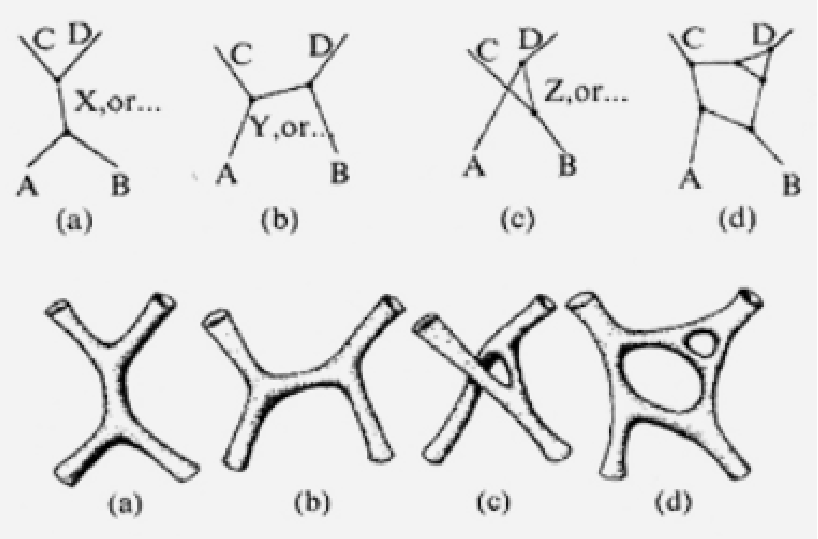


Cobordisms

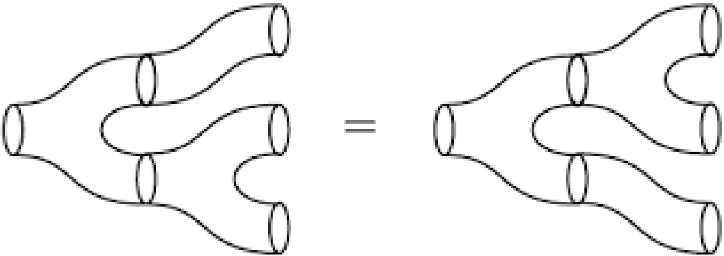


A cobordism $\emptyset \to \emptyset$.

Feynman diagrams



Associativity

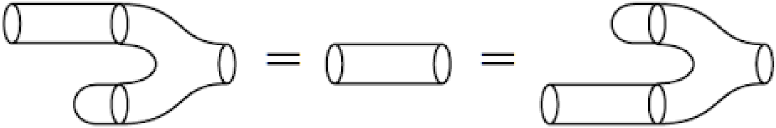


$$(a + b) + c = a + (b + c)$$

Commutivity

$$a + b = b + a$$

Neutrality



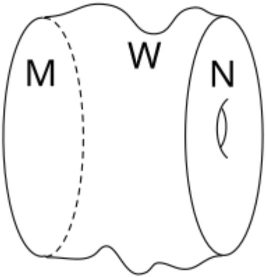
$$a + 0 = a = 0 + a$$

TQFT

A *topological quantum field theory* Z assigns to every cobordism a linear map between vector spaces:

$$\text{cobordism}: \qquad W: M \to N$$

$$\text{linear map}: \qquad ZW: ZM \to ZN$$



TQFT

Some properties of a TQFT hold:

- A TQFT assigns the empty set $\emptyset$ the real numbers $\mathbb{R}$:

$$Z\emptyset \cong \mathbb{R}.$$

- A TQFT assigns the cylinder $M \times [0,1]$, a cobordism $M \to M$, the identity:

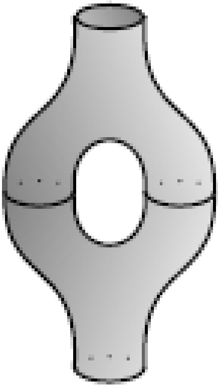
$$Z(M \times [0,1]) = \mathrm{id}_{ZM}\,.$$

- For cobordisms $V\colon L \to M$ and $W\colon M \to N$, one has linear maps $ZV\colon ZL \to ZM$ and $ZW\colon ZM \to ZN$ with:

$$Z(V \circ W) = ZV \circ ZW.$$

TQFT

A TQFT can be *hole blind*:



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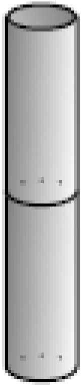


A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with:

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$ is given by multiplication with a value $a \in \mathbb{R}$, hence $f(x) = ax$ for all $x \in \mathbb{R}$.

Putting $y = 0$ yields:

$$f(x) = f(x+0) = f(x) + f(0),$$

hence $f(0) = 0$.

Putting $y = -x$ yields:

$$f(x) + f(-x) = f(x-x) = f(0) = 0,$$

hence $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

One has:

$$f(n \cdot x) = f(\underbrace{x + \ldots + x}_{n \text{ times}}) = \underbrace{f(x) + \ldots + f(x)}_{n \text{ times}} = n \cdot f(x)$$

One has:

$$n \cdot f\left(\frac{x}{n}\right) = f\left(n \cdot \frac{x}{n}\right) = f(x) \Rightarrow f\left(\frac{x}{n}\right) = \frac{1}{n} f(x)$$

Summarizing all previous results yields $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$.

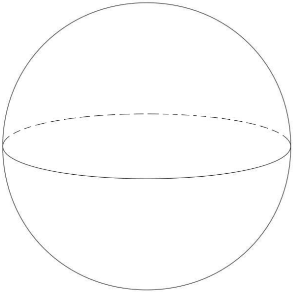
Let $r \in \mathbb{R}$ and $(q_n)_{n \in \mathbb{N}} \subset \mathbb{Q}$ with $r = \lim_{n \to \infty} q_n$, then:

$$f(rx) = f(\lim_{n \to \infty} q_n x) = \lim_{n \to \infty} f(q_n x) = \lim_{n \to \infty} q_n f(x) = rf(x)$$

Hence:

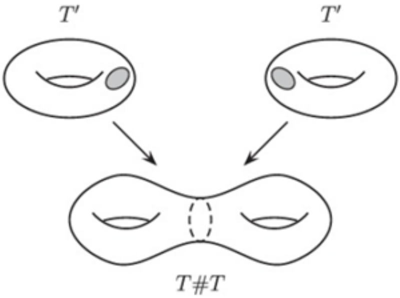
$$f(x) = f(x \cdot 1) = x \cdot \underbrace{f(1)}_{=a}$$

Topological invariants



The sphere $S\colon \emptyset \to \emptyset$ becomes a linear map $ZS\colon \mathbb{R} \to \mathbb{R}$, uniquely described by its value $ZS(1)$.

Topological invariants



Other figures yield other topological invariants, which often can be computed seperating it in parts and calculating their topological invariants.

Thanks for your attention! :-)

Questions?